In memory of Pieter Hofstra

Toposes and C^*-algebras [1]

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Overview and motivation

1

How is polar decomposition from operator theory interpretated in topos theory?

2

What is the common ground shared by toposes and C^*-algebras? How do we match concepts between the two disciplines?

3

We define a left-cancellative category and a topos of a C^*-algebra in a manner that resembles what is done in pseudogroup and inverse semigroup theory [2, 3], while recognizing that for C^*-algebras there are some distinct and novel points of departure from the semigroup constructions.

Overview and motivation continued

4

We work under a certain hypothesis we call a supported C^*-algebra.

5

The topos interpretation of polar decomposition we shall see is part of a correspondence between quotients of a torsion-free generator of the topos of a C^*-algebra and certain subcategories of its left-cancellative category.

Example: bounded operators on Hilbert space

Support/cosupport

Let $\mathcal{H}$ denote a Hilbert space
Let $B(\mathcal{H})$ denote the C^*-algebra of bounded operators on $\mathcal{H}$.
$\forall\, S, T, R \in B(\mathcal{H})\;\; \mathrm{Ker}(S) \subseteq \mathrm{Ker}(T) \Rightarrow \mathrm{Ker}(SR) \subseteq \mathrm{Ker}(TR)$.

For $T \in B(\mathcal{H})$ let $N(T)$ denote the projection associated with the subspace $\mathrm{Ker}(T)$.
$\forall\, S, T, R\;\; N(S) \leq N(T) \Rightarrow N(SR) \leq N(TR)$.

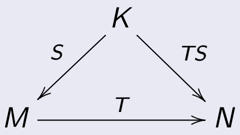
The support projection $C(T) = I - N(T^*)$ is the projection associated with $\overline{\mathrm{Ran}(T)}$.
$\forall\, S, T, R\; :\;\; C(S) \leq C(T) \Rightarrow C(RS) \leq C(RT)$

Example continued

A category associated with $\mathcal{H}$

Let $L(\mathcal{H})$ denote the following category.
Objects: the subspaces of $\mathcal{H}$.
Morphisms: $T : M \rightarrow N$ is a linear operator T on $\mathcal{H}$ such that $\mathrm{Ker}(T) = M^{\perp}$, and $\mathrm{Ran}(T) \subseteq N$.

$L(\mathcal{H})$ is a category



Example continued

We must have $\mathrm{Ker}(TS) = K^\perp$

We have $\mathrm{Ker}(S) = K^\perp$ and $\mathrm{Ran}(S) \subseteq M = \mathrm{Ker}(T)^\perp$
Therefore, $\mathrm{Ker}(T) \subseteq \mathrm{Ker}(S^*)$

Hence, $\mathrm{Ker}(TS) \subseteq \mathrm{Ker}(S^*S) \overbrace{=}^{exercise} \mathrm{Ker}(S)$
The other inclusion $\mathrm{Ker}(S) \subseteq \mathrm{Ker}(TS)$ is trivial.
Therefore, $\mathrm{Ker}(TS) = \mathrm{Ker}(S)$

$L(\mathcal{H})$ is left-cancellative

Let $T : M \rightarrow N$ be a morphism. Let P denote the projection associated with the subspace M: $\mathrm{Ker}(T) = \mathrm{Ker}(P)$.
Suppose that $TS = TR$, where $S, R : K \rightarrow M$.
Then for any $v \in \mathcal{H}$, we have $S(v) - R(v) \in \mathrm{Ker}(T)$.
Thus, $P(S(v) - R(v)) = 0$, whence
$S(v) = PS(v) = PR(v) = R(v)$. Thus, $S = R$.

Supported C^*-algebra $\mathcal{A}$

Support/cosupport projection

Let $T \in \mathcal{A}$.

- A support projection $C(T)$ satisfies $C(T) \leq P$ iff $T = PT$ (so $T = C(T)T$)
- A cosupport projection $N(T)$ satisfies $P \leq N(T)$ iff $TP = 0$ (so $TN(T) = 0$)

Lemma: If $C(T)$ exists, then $C(TT^*) = C(T)$.

This follows from the C^*-identity $\|TT^*\| = \|T\|^2$

Supported C^*-algebra $\mathcal{A}$ continued

Support hypothesis

We shall say that a C^*-algebra $\mathcal{A}$ *is supported* if:

1. every $T \in \mathcal{A}$ has a support projection $C(T)$ such that
2. $\forall\, S, T, R \,:\; C(S) \leq C(T) \Rightarrow C(RS) \leq C(RT)$ (Stability).

The support hypothesis has an equivalent cosupport form:

Cosupport

1. every T has a cosupport projection $N(T)$ such that
2. $\forall\, S, T, R \,:\; N(S) \leq N(T) \Rightarrow N(SR) \leq N(TR)$.

von Neumann algebra

$B(\mathcal{H})$ and more generally any von Neumann algebra is supported in this sense.

Polar decomposition

Existence

$T = VA$ such that:

1. V is a partial isometry: $VV^*V = V$
2. A is positive: self-adjoint and spectrum $\subseteq [0, \infty)$
3. $C(A) = V^*V$

Note: $T^*T = AV^*VA = AC(A)A = AA = A^2$; $|T| = \sqrt{T^*T} = A$, so that $C(T^*) = C(T^*T) = C(A^2) = C(A) = C(|T|)$

Another way: $T = V|T|$; $C(T^*) = V^*V$

Polar decomposition continued

Uniqueness

If $T = VA = UB$; $C(A) = V^* V$; $C(B) = U^* U$
then $U = V$ and $A = B$.

Supported implies uniqueness

If $\mathcal{A}$ is supported, then a polar decomposition of an element is necessarily unique.

The left-cancellative category $L(\mathcal{A})$

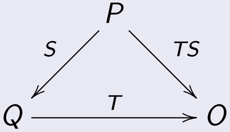
Definition of $L(\mathcal{A})$

Let $\mathcal{A}$ denote a unital supported C^*-algebra.

Objects: projections P of $\mathcal{A}$ ($P^*P = P$)
Morphisms: $T : P \rightharpoonup Q$, $C(T^*) = P$ (iff $N(T) = I - P$),
and $T = QT$ ($C(T) \leq Q$)

Another way: a morphism is a pair (T, Q) such that $T = QT$.
Domain of (T, Q) is $C(T^*)$
Codomain of (T, Q) is Q

$L(\mathcal{A})$ is a category



We have $C(S) \leq Q = C(T^*)$.

Then $P = C(S^*) = C(S^*S) \overbrace{\leq}^{\text{stability}} C(S^*T^*) \leq C(S^*)$
Thus, $P = C(S^*T^*) = C((TS)^*)$.
We also have $T = OT$ so of course $TS = OTS$.

The identity morphism $P \to P$ is simply P.
Indeed, if $T : P \to Q$ is a morphism,
then $TP = TC(T^*) = T$ and $QT = T$.

$L(\mathcal{A})$ is left-cancellative

Suppose that we have morphisms

$$P \overset{S}{\underset{R}{\rightrightarrows}} Q \xrightarrow{\ T\ } O \quad \text{such that } TS = TR\,.$$

Then $T(S-R) = 0 \Rightarrow (S^* - R^*)T^* = 0$

$\Rightarrow C((S^* - R^*)T^*) = 0\,.$

We have $C(Q) = Q = C(T^*)$

Therefore, $C((S^* - R^*)Q) \overbrace{\leq}^{\text{stability}} C((S^* - R^*)T^*) = 0$

$\Rightarrow (S^* - R^*)Q = 0$

$\Rightarrow Q(S-R) = 0 \Rightarrow S = QS = QR = R\,.$

Topos of presheaves on $L(\mathcal{A})$: $\mathscr{B}(\mathcal{A})$

Definition of $\mathscr{B}(\mathcal{A})$

An object of this topos is a functor:
$F : L(\mathcal{A})^{\mathrm{op}} \longrightarrow Set$

Representable presheaf

Let Q be a projection.
$Q : L(\mathcal{A})^{\mathrm{op}} \longrightarrow Set$
$Q(P) = L(\mathcal{A})(P, Q) = \{ T \in \mathcal{A} \mid C(T^*) = P \,;\; T = QT \}$
Transition in Q along $S : O \rightarrow P$: $T \cdot S = TS$ for $C(T^*) = P$

$\mathscr{B}(\mathcal{A})$ continued

Representable presheaf associated with the unit I

$I : L(\mathcal{A})^{\mathrm{op}} \longrightarrow Set$

$I(P) = \{ T \in \mathcal{A} \mid C(T^*) = P \}$

Transition in I along $S : O \to P$: $T \cdot S = TS$ for $C(T^*) = P$

(Existence of unit I not necessary)

$\mathscr{B}(\mathcal{A})$ is an étendue

The presheaf I is a torsion-free generator [4].

The positive quotient

The presheaf of positive operators

$I^+ : L(\mathcal{A})^{\mathrm{op}} \longrightarrow Set$
$I^+(P) = \{ A \in \mathcal{A} \mid 0 \leq A \,;\; C(A) = P \}$
Transition in I^+ :
let $S : P \rightarrow Q$ is a morphism of $L(\mathcal{A})$ and $C(A) = Q$
Define $A \cdot S = S^* A S = (\sqrt{A} S)^* \sqrt{A} S$, which is positive.
Then $C(S^* A S) = C((\sqrt{A} S)^* \sqrt{A} S) = C((\sqrt{A} S)^*) = P$, where
$\sqrt{A} : Q \rightarrow Q$ is a morphism of $L(\mathcal{A})$; $C(\sqrt{A}) = C(A) = Q$

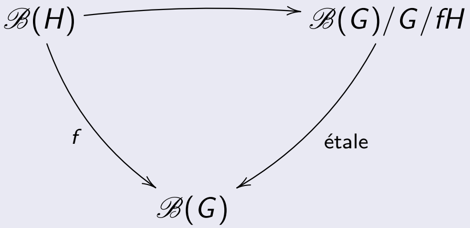
The quotient map $d : I \longrightarrow I^+$

$d_P : I(P) \rightarrow I^+(P) \,;\; d_P(T) = T^* T$
d is a natural transformation: $S^* T^* T S = (TS)^* TS$
d is an epimorphism: if $C(A) = P$, then $d_P(\sqrt{A}) = A$.
Caution: $A \mapsto \sqrt{A}$ is not a section of d .

Wide subcategory

Group actions

Suppose that $f : H \to G$ is an injective homomorphism.
Then the (right) coset G/fH is a G-set (object of $\mathscr{B}(G)$), and $G \to G/fH$ is an equivariant map (morphism of $\mathscr{B}(G)$).
We have geometric morphisms:



The one depicted horizontally is an equivalence.
Therefore, the one associated with f is étale.

Wide subcategory of $L(\mathcal{A})$

A functor $\mathcal{D} \longrightarrow L(\mathcal{A})$ is *a wide subcategory* if:

1. $\mathcal{D}$ has the same set of objects as $L(\mathcal{A})$, which is the set of projections of $\mathcal{A}$;
2. the functor is faithful - usually we just assume $\mathcal{D}(P, Q) \subseteq L(\mathcal{A})(P, Q)$, for every P, Q;
3. every subprojection $P \leq Q$ is a morphism of $\mathcal{D}$. Thus, for all projections P, Q we have $\mathcal{P}(\mathcal{A})(P, Q) \subseteq \mathcal{D}(P, Q) \subseteq L(\mathcal{A})(P, Q)$;
4. for $S, T \in \mathcal{A}$ such that $C(T) \leq C(S^*)$ $(T = C(S^*)T)$, if $S, ST \in \mathcal{D}$, then $T \in \mathcal{D}$.

Examples of wide subcategories of $L(\mathcal{A})$

Two trivial ones

$\mathcal{P}(\mathcal{A}) \longrightarrow L(\mathcal{A})$ and $L(\mathcal{A}) \longrightarrow L(\mathcal{A})$

The wide subcategory of partial isometries

$\partial(\mathcal{A}) \longrightarrow L(\mathcal{A})$

$V : P \rightarrow Q$ such that $P = V^*V$ and $V = QV$

Right cosets of a wide subcategory $\mathcal{D} \longrightarrow L(\mathcal{A})$

The right coset of $T \in \mathcal{A}$

$\mathcal{D}T = \{\, ST \mid S \in \mathcal{D}\,;\; C(T) \leq C(S^*) \,\}$

The presheaf of right cosets

Define a presheaf
$I/\mathcal{D}\,(P) = \{\, \mathcal{D}T \mid C(T^*) = P \,\}$
Transition along $S : P \rightarrow Q$ is given by $\mathcal{D}T \cdot S = \mathcal{D}(TS)$.

The quotient of right cosets

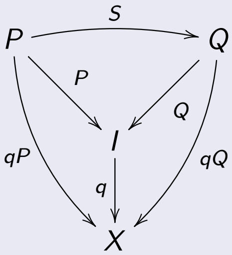
$q : I \rightarrow I/\mathcal{D}$
$q_P : I(P) \rightarrow I/\mathcal{D}\,(P)\,;\; q_p(T) = \mathcal{D}T$, for $C(T^*) = P$

The principal fiber of a map $I \rightarrow X$

Given a map $q : I \rightarrow X$ of $\mathscr{B}(\mathcal{A})$

Define a subcategory $\mathcal{F}(q) \longrightarrow L(\mathcal{A})$:
Objects: projections of $\mathcal{A}$
Morphisms: $S : P \rightarrow Q$ such that $q_P(S) = q_P(P)$,
where $q_P : I(P) \rightarrow X(P)$.

Morphism of $\mathcal{F}(q)$ interpreted in $\mathscr{B}(\mathcal{A})$



$\mathcal{F}(q) \longrightarrow L(\mathcal{A})$ is wide

PROOF: 4. Suppose we have $T : O \rightarrow P$ and $S : P \rightarrow Q$, such that $q_O(ST) = q_O(O)$ and $q_P(S) = q_P(P)$.
Then we have

$$q_O(T) = q_O(PT) = q_P(P) \cdot T = q_P(S) \cdot T = q_O(ST) = q_O(O)$$

Example: the wide subcategory of the positive quotient

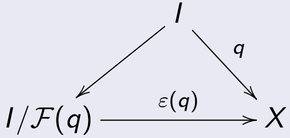
Proposition

The principal fiber of the positive quotient $d : I \twoheadrightarrow I^+$ coincides with the wide subcategory of partial isometries $\partial(\mathcal{A}) \longrightarrow L(\mathcal{A})$.

The counit

Start with $q : I \twoheadrightarrow X$

Then form $\mathcal{F}(q) \longrightarrow L(\mathcal{A})$, and its quotient of cosets.



The component $\varepsilon(q)_P$ at a projection P of the factoring map $\varepsilon(q)$ is defined by $\varepsilon(q)_P(\mathcal{F}(q)T) = q_P(T)\ ;\ C(T^*) = P$

Exact quotient of I

We say that $q : I \twoheadrightarrow X$ is exact if $\varepsilon(q)$ is an isomorphism.

The bijective correspondence

Definition

A wide subcategory $\mathcal{D} \longrightarrow L(\mathcal{A})$ *is principal* if for all $S \in \mathcal{D}$ we have $C(S^*) \in \mathcal{D}S$.

Remark

A wide subcategory $\mathcal{D} \longrightarrow L(\mathcal{A})$ is principal iff for all $S \in \mathcal{D}$ we have $\mathcal{D}S = \mathcal{D}C(S^*)$.

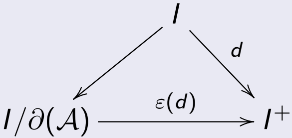
Proposition

There is a bijective correspondence between the principal wide subcategories of $L(\mathcal{A})$, and the exact quotients of I in $\mathscr{B}(\mathcal{A})$.

Polar decomposition in $\mathscr{B}(\mathcal{A})$

Theorem

Let $\mathcal{A}$ be a unital supported C^*-algebra.
Then $\mathcal{A}$ has polar decomposition iff the positive quotient d is exact.



Corollary

The positive quotient in the topos of a von Neumann algebra is exact.

Proof: A von Neumann algebra has polar decomposition.

'Scratching the surface'

Morita equivalence

Cohomology

Factor theory

Factor theory of von Neumann algebras is related to isotropy theory of toposes.

Topos representations of a (supported) C^*-algebra $\mathcal{A}$

This is a functor

$$L(\mathcal{A}) \longrightarrow \mathscr{E} \ ,$$

which may be filtered, etc.
For instance, the canonical one

$$Yoneda : L(\mathcal{A}) \longrightarrow \mathscr{B}(\mathcal{A}) \ .$$

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Thank you